Axial anomaly at finite temperature

1985 J. Phys. A: Math. Gen. 18 L963

(http://iopscience.iop.org/0305-4470/18/15/011)

J. Phys. A: Math. Gen. 18 (1985) L963-L967. Printed in Great Britain

LETTER TO THE EDITOR

Axial anomaly at finite temperature

S Chaturvedi, Neelima Gupte and V Srinivasan
School of Physics, University of Hyderabad, Hyderabad 500134, India

Received 30 May 1985

Abstract. The Jackiw-Bardeen-Adler anomaly for QED_4 and QED_2 are calculated at finite temperature. It is found that the anomaly is independent of temperature. Ishikawa's method for calculating the quantised Hall effect is extended to finite temperature.

Calculations in quantum field theory (QFT) at finite temperature have acquired significance especially with respect to the early universe. Since the axial anomaly occupies a special place in QFT it would be natural to ask what happens to it at finite temperature. Zhao-Won-Yun has calculated the QED_4 triangle diagram at finite temperature (Zhao-Won-Yun 1984) and has shown that the anomaly is not affected at non-zero temperature. Reuter and Dittrich have calculated QED_2 at finite temperature and show that it is unaffected (Dittrich and Reuter 1984). In this letter we calculate the axial anomaly at finite temperature extending Fujikawa's (1980) method. We confirm that the axial anomaly is indeed not affected: intuitively one expects this result as the anomaly arises from topological considerations and is related to the Atiyah-Singer index. Also one knows that in superconductivity the flux quantisation does not depend on temperature.

The Schwinger model in $(2+1)$ dimensions has acquired significance, since the integral quantised Hall effect can be explained on the basis of this model (Ishikawa 1984). Although one knows that in odd spacetime dimensions the anomaly is absent, Ishikawa has shown that something similar to it happens in $(2+1)$ dimensions; we extend his arguments to finite temperature and show that indeed the integral quantised Hall effect does not depend on temperature, i.e. the integrally quantised conductivity is independent of temperature.

Consider a SU(N) Yang-Mills field coupled with fermions whose Lagrangian is

$$\mathscr{L} = \bar{\psi} i \gamma^\alpha D_\alpha \psi - m\bar{\psi}\psi + (1/2g^2) F^{\mu\nu} F_{\mu\nu}.$$

We consider this in Euclidean spacetime by replacing $x^0 \to -ix^4$, $A_0 \to iA_4$. After this operation

$$\not{D} = \nu^4 D_4 + \gamma^k D_k = \gamma^\alpha(\partial_\alpha + A_\alpha).$$

The γ matrices follow the Bjorken-Drell convention, $\gamma_5 = -\gamma^1\gamma^2\gamma^3\gamma^4$ and γ^5 is Hermitian. The eigenvalues equation for $\not{D}$ is

$$i\not{D}\phi_n = \lambda_n \phi_n.$$

The generating functional of the complete Green function is given by

$$z(\eta, \bar{\eta}, J_\mu) = \frac{1}{N} \int d\mu \, \exp \int (\mathscr{L} + \eta\bar{\psi} + \bar{\psi}\eta - J_\mu A_\mu)\, dx.$$

0305-4470/85/150963+05\$02.25

Here

$$d\mu = \prod_x \mathscr{D}A_\mu(x)\, d\bar{\psi}\, d\psi$$

where $\mathscr{D}A_\mu(x)$ contains the Faddeev–Popov factors. Under a local chiral transformation

$$\psi(x) \to \psi'(x') \equiv \exp(i\alpha(x)\gamma_5)\psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}(x') = \bar{\psi}(x)\exp(i\alpha(x)\gamma_5).$$

The Jacobian factor for $d\bar{\psi}$ and $d\psi$ leads to the following change in $d\mu$ due to the chiral transformations

$$d\mu \to d\mu \exp\left(-2i \int dx\, \alpha(x) A(x)\right)$$

where

$$A(x) = \sum_n \phi_n^+(x)\gamma_5\phi_n(x)$$

where the ϕ_n's are eigenfunctions of the operator $\not{D}$. $A(x)$ is the definition of the anomaly in Fujikawa's method. It is evaluated by Fujikawa in the following way using a regulator

$$A(x) = \lim_{M\to\infty} \lim_{x\to y} \operatorname{Tr} T_5 \exp(-\not{D}^2/M^2)\, \delta^4(x-y)$$
$$= \lim_{M\to\infty} \operatorname{Tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 \exp(-\not{D}^2/M^2)\exp[ik(x-y)].$$

To proceed to the finite temperature we replace (Ezawa *et al* 1957, Jackiw and Dolan 1974)

$$\int d^4x \to \int_0^\beta dx^0 \int_{-\infty}^{+\infty} d^3x$$

with

$$A_\mu(x^0+\beta, \boldsymbol{x}) = A_\mu(x_\phi, \boldsymbol{x})$$

and

$$\psi(x_0, \beta, \boldsymbol{x}) = -\psi(x_0, \boldsymbol{x}).$$

Then $A(x)$ becomes

$$A(x) = \lim_{M\to\infty} \operatorname{Tr} \int \frac{d^4k}{(2\pi)^4} \frac{2\pi}{\beta} \sum_n \delta\left(k_4 - \frac{2\pi}{\beta}(n+\tfrac{1}{2})\right) \exp\left(-\frac{\not{D}^2}{M^2}\right) \exp[ik(x-y)]$$
$$= \lim \operatorname{Tr}\left(\gamma_5 \frac{[\gamma_\mu, \gamma_\nu]}{2} F_{\mu\nu}\right)^2 \left(\frac{1}{2M^2}\right)^2 \frac{1}{2\pi\beta} \sum \exp(-\omega_n^2/M^2) \times I$$

where

$$\omega_n = (2\pi/\beta)(n+\tfrac{1}{2})$$

and

$$I = \int \frac{d^3k}{(2\pi)^3} \exp[-(k_1^2 + k_2^2 + k_3^2)].$$

Now we have to compute the term independent of M, as the rest vanish in the limit $M\to\infty$. To do this we expand $\Sigma \exp-(\omega_n^2/M^2)$ as an Euler–Maclaurin series

$$\sum \exp(-\omega_n^2/M^2)=\frac{\sqrt{\pi}}{2\pi}\beta M(1+\mathrm{O}(M)).$$

Thus we find the anomaly to be

$$A(x)=-\frac{1}{16\pi^2}\operatorname{Tr}{}^*F^{\mu\nu}F_{\mu\nu}$$

which is the same as its zero temperature value.

Consider a system described by the Lagrangian in (2+1) dimensions.

$$\mathscr{L}=\bar{\psi}(\gamma^\mu)(\mathrm{i}D_\mu+eA_\mu)\psi+\bar{\psi}\psi U(x).$$

Here ψ stands for the fermion field, $A_\mu(x)$ the external static electromagnetic field, $U(x)$ is a static background field and the γ_μ's are 2×2 matrices satisfying

$$\varepsilon_{\mu\nu\rho}\gamma_\mu\gamma_\nu\gamma_\rho=1.$$

The generating functional is

$$z=\int \mathrm{d}\mu\ \exp\left(-\mathrm{i}\int \mathscr{L}\,\mathrm{d}^3x\right)$$

where

$$\mathrm{d}\mu=\mathrm{d}\bar{\psi}\,\mathrm{d}\psi.$$

Consider the following analogue of the chiral transformation

$$\psi(\bar{x},t)=\exp(\mathrm{i}\alpha(x)\gamma_0)\psi(\bar{x},t)$$
$$\bar{\psi}(\bar{x},t)=\bar{\psi}(x,t)\exp(\mathrm{i}\alpha(x)\gamma_0).$$

The Jacobian factor for $\mathrm{d}\bar{\psi}\,\mathrm{d}\psi$ becomes

$$\mathrm{d}\mu\to \mathrm{d}\mu'\exp\left(-2\mathrm{i}\int \mathrm{d}x\,\alpha(x)A(x)\right)$$

where

$$A(x)=\sum_n \phi_n^+(x)\gamma_0\phi_n(x).$$

If $A(x)$ is evaluated at finite temperature by the procedure given in the previous section one finds

$$A(x)=-(1/2\pi)eF_{12}.$$

Thus the anomaly in (2+1) dimensions does not change with temperature. If $\alpha(x)=\alpha$, a constant, the change in the phase factor due to the change in measure is

$$4\pi\frac{e}{h}\int \mathrm{d}x\,F_{12}(x).$$

Demanding that the action be unaltered under the transformations considered leads to

$$4\pi \frac{e}{h} \int \mathrm{d}x\, F_{12} = 2\pi \times (\text{integer})$$

therefore

$$\int \mathrm{d}x\, F_{12} = \frac{h}{2e} \times (\text{integer}).$$

Since the anomaly term is the same at all temperatures we conclude that the flux quantisation does not depend on temperature. Let us consider the anomaly term's contribution to the ground state of the system. It is equal to $-\Sigma\lambda_n$.

If we change A_0 adiabatically, then this will change to $-\Sigma\lambda_n + \delta\lambda_n$ where

$$\delta\lambda_n = \phi_n^+ \gamma^0 e \delta A_0 \phi_n$$

and $\Sigma\delta\lambda_n$ can be easily shown to be

$$\frac{e^2}{h} \int \delta A_0(x) F_{ij}(x)\, \mathrm{d}x.$$

The total increase in energy is given by

$$\Delta E = \frac{e^2}{h} \int \mathrm{d}x\, A_0(x) F_{ij}(x).$$

In the case of the Hall effect a constant electrical field is applied in the y direction and a magnetic field is applied orthogonal to the (x, y) plane. For such a system

$$A_0 = E_y \qquad F_{ij} = H.$$

The Hall current is given by

$$\begin{aligned} I_x &= \int \mathrm{d}y \frac{\delta}{\delta A_x(x, y)} \frac{e^2}{h} \int A_0(x) F_{ij}(x)\, \mathrm{d}x \\ &= \int \mathrm{d}y \frac{e^2}{h} \partial_y A_0(x) = \frac{e^2}{h} V. \end{aligned}$$

We find that

$$I = (e^2/h) V$$

is unaltered at finite temperature. Thus the integral quantised Hall effect does not change with temperature.

Thus we have shown that the anomaly is not affected by temperature. As a consequence we see that the integral quantised Hall effect is unaltered at finite temperature. However the fractional Hall effect could depend on temperature as it is difficult to correlate that with the anomaly.

Dittrich and Reuter have calculated the axial anomaly in QED_2 by the ξ function technique at finite temperature. Extending Fujikawa's method we also find very easily that it is

$$A(x) = -\frac{1}{2\pi} \varepsilon_{\mu\nu} F^{\mu\nu}.$$

It will be interesting to investigate the effect of charge fractionalisation due to the anomaly at finite temperature. Such a study is in progress and will be reported elsewhere.

One of us (NG) wishes to thank the CSIR for financial support. We thank Dr Anishetty for discussions and a critical reading of the manuscript.

References

Dittrich W and Reuter M 1984 *Preprint* University of Tubingen
Ezawa H, Tomozawa Y and Umezawa H 1957 *Nuovo Cimento* **5** 810
Fujikawa K 1980 *Phys. Rev.* **21** 2848
Ishikawa K 1984 *Phys. Rev. Lett.* **53** 1615
Jackiw R and Dolan L 1974 *Phys. Rev.* **9** 3320
Zhao-Won-Yun 1984 *Preprint* Beijing